JOURNAL OF APPROXIMATION THEORY **24**, 35–50 (1978)

Exact Comonotone Approximation

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Communicated by Oved Shisha

Received February 7, 1977

1. INTRODUCTION

Let N, N^+ and R denote, respectively the set of positive integers, the set of nonnegative integers, and the set of real numbers; let Π_n, $n \in N$, denote the set of algebraic polynomials of degree less than or equal to n; let $C[a, b]$ denote the set of real continuous functions on $[a, b]$; let $\omega(f; \cdot)$ denote the modulus of continuity of f; let $\|f\|$ denote the sup norm of $f \in C[a, b]$; and, finally, let $c_1, c_2, \ldots, d_1, d_2, \ldots$, denote absolute positive constants.

In this paper, we are interested in determining how well we can approximate continuous functions which increase and decrease a finite number of times on a closed interval by polynomials which share the same monotonicity properties on this interval.

A function $f \in C[a, b]$ is called *piecewise monotone* if it has a finite number only of relative maxima and minima on $[a, b]$. The points a and b together with the relative maxima and minima of f are called the *peaks* of f. Given that f is a piecewise monotone function on $[-1, 1]$ with peaks at the points

$$-1 = \xi_0 < \xi_1 < \cdots < \xi_k = 1,$$

we define the degree of comonotone approximation to f by elements of Π_n to be

$$E_n^*(f) = \inf\{\|f - p\| : p \in \Pi_n(f)\},$$

where $\Pi_n(f)$ is the set of elements of Π_n which have the same monotonicity as f on each of the subintervals (ξ_i, ξ_{i+1}), $i = 0, 1, \ldots, k - 1$.

* The research of this author was partially supported by a Temple University Summer Research Grant.

0021-9045/78/0241-0035$02.00/0

If $k = 1$, that is, if f is a monotone function on $[-1, 1]$, then Lorentz and Zeller [3] have shown that

$$E_n^*(f) \leq c\omega(f; n^{-1}), \qquad n = 1, 2, \ldots;$$

that is, we can approximate monotnely with the same order of approximation guaranteed by Jackson's theorem for

$$E_n(f) = \inf\{\|f - p\| : p \in \Pi_n\}.$$

Ideally, we would like to show that

$$E_n^*(f) = O(\omega(f; n^{-1})) \tag{1}$$

for any piecewise monotone function f. The quantitative results already known on comonotone approximation are of two kinds—monotonicity is preserved, but at a loss in degree of approximation, or the Jackson order of approximation is retained, but comonotonicity is lost around the peaks of f. More specifically, regarding exact comonotone approximation (where monotonicity is preserved even around the peaks), it has been shown by Passow and L. Raymon [6] that if f is a piecewise monotone function on $[-1, 1]$, then, given any $\epsilon > 0$,

$$E_n^*(f) = o(\omega(f; n^{-1+\epsilon})). \tag{2}$$

In this paper we obtain a Jackson-type theorem for comonotone approximation. In particular, we obtain estimates of the form (1) for comonotone approximation to piecewise monotone functions of a certain type. This class of functions will include, in particular, all piecewise linear functions which are non-constant in each subsegment containing a local extremum.

The DeVore kernel is defined to be $V_n(t)$, where

$$V_n(t) = v_n \left[\frac{P_{2n}(t)}{(t^2 - \alpha_n{}^2)}\right]^2,$$

where $P_{2n}(t)$ is the Legendre polynomial of degree $2n$, α_n is its smallest positive zero, and v_n is a normalizing constant chosen so that

$$\int_{-1}^{1} V_n(t)\, dt = 1.$$

If we vary the DeVore kernel by dividing out additional zeros of $P_{2n}(t)$, that is, if we define, for each $j \in N$, the $D - j$ kernel $V_{n,j}(t)$ by

$$V_{n,j}(t) = v_{n,j} \left[\frac{P_{2n}(t)}{(t^2 - \alpha_{1,n}^2) \cdots (t^2 - \alpha_{j,n}^2)}\right]^2, \tag{3}$$

where $P_{2n}(t)$ is again the Legendre polynomial of degree $2n$, $\alpha_{1,n}, \alpha_{2,n},\ldots, \alpha_{j,n}$ are the j smallest positive zeros of $P_{2n}(t)$, and $v_{n,j}$ is a normalizing constant chosen so that

$$\int_{-1}^{1} V_{n,j}(t)\, dt = 1,$$

then we obtain a whole sequence of algebraic kernels with the following property: If we define the nth $D - j$ polynomial of $f \in C[-\frac{1}{2}, \frac{1}{2}]$ to be

$$Q_{n,j}(f; x) = \int_{-1/2}^{1/2} V_{n,j}(t - x) f(t)\, dt, \tag{4}$$

then $Q_{n,j}(f; x) \in \Pi_{4(n-j)}$ for each $n \geqslant j$. It is shown in [1, Chap. 6], that if $0 < \delta < \frac{1}{2}$, then

$$\|f - Q_{n,j}(f)\|_{[-\delta,\delta]} \leqslant c_\delta \omega(f; n^{-1}), \tag{5}$$

where c_δ is a constant depending only on j and δ.

In proving our main result, we will need to use one of the $D - j$ kernels for which $j \geqslant 2$. We will, in fact, choose to work with the $D - 2$ kernel.

2. Preliminaries

Note. We adopt the following convention regarding notation: Suppose we are given a "distinguished" set of points $\{t_i\}$ $(i = 1,\ldots, r)$ satisfykng

$$a < t_1 < t_2 < \cdots < t_r < b.$$

Then, when we refer to the number d with respect to the set $\{a, b, t_i\}$, we will always mean the number $\frac{1}{2} \min_i \{(t_1 - a), (b - t_r), (t_{i+1} - t_i)\}$, and if $0 < \epsilon < d$, we denote by S_ϵ, the collection of intervals

$$\{(a, t_1 - \epsilon), (t_r + \epsilon, b), (t_i + \epsilon, t_{i+1} - \epsilon)\} \qquad (i = 1,\ldots, r - 1).$$

Before stating the theorem, we will make the following definitions and observations:

Definition 1. Let $f(x)$ be a piecewise monotone function on $[a, b]$, with peaks at

$$a = \xi_0 < \xi_1 < \cdots < \xi_{p+1} = b,$$

where $p \geqslant 1$. We say that $f(x)$ satisfies a *convexity condition* (around its peaks) if there exists $\epsilon > 0$ such that in each of the intervals $(\xi_i - \epsilon, \xi_i + \epsilon)$, $i = 1, 2,\ldots, p$,

$$\Delta^2 f(x) \geqslant 0 \text{ if } \xi_i \text{ is a local minimum,}$$
$$\Delta^2 f(x) \leqslant 0 \text{ if } \xi_i \text{ is a local maximum.}$$

Note 1. The following statements are true for any function f satisfying the above definition (see [7, pp. 108-109]):

(i) The right- and left-hand derivatives of f exist at each point $x \in (\xi_i - \epsilon, \xi_i + \epsilon)$, $i = 1,..., p$ (and are equal almost everywhere) and are denoted by $D^+f(x)$ and $D^-f(x)$, respectively.

(ii) The functions $D^+f(x)$ and $D^-f(x)$ are non-decreasing [non-increasing] in the intervals $(\xi_i - \epsilon, \xi_i + \epsilon)$ for which ξ_i is a local minimum [local maximum], and if $x, y \in (\xi_i - e, \xi_i + \epsilon)$ and $x < y$, then $D^+f(x) \leqslant D^-f(y) \leqslant D^+f(y)[D^+f(x) \geqslant D^-f(y) \geqslant D^+f(y)]$.

DEFINITION 2. Let f be as in Definition 1. Then f is *properly piecewise monotone* if it satisfies a convexity condition for some $\epsilon > 0$ and if, for each $i = 1,..., p$,

$$|D^+f(\xi_i)| > 0, \; |D^-f(\xi_i)| > 0.$$

Note 2. The following two observations follow easily from the definition of "properly piecewise monotone" together with Note 1:

(i) Define, for each $i = 1,..., p$,

$$M_i = \max\{|D^-f(\xi_i)|, |D^+f(\xi_i)|\}.$$

Then for each such i, there exists $0 < \epsilon_i < \epsilon$ such that

$$|f'(x)| \leqslant 2M_i \text{ a.e.} \qquad \text{in } (\xi_i - \epsilon_i, \xi_i + \epsilon_i). \tag{6}$$

(ii) Define, for each $i = 1,..., p$,

$$m_i = \min\{|D^-f(\xi_i)|, |D^+f(\xi_i)|\}.$$

Then for each such i,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \geqslant m_i \tag{7}$$

whenever x and y belong to one of the intervals $(\xi_i - \epsilon, \xi_i)$ or $(\xi_i, \xi_i + \epsilon)$, $x \neq y$.

We are now ready to state the main result:

THEOREM. *Let $f(x)$ be a properly piecewise monotone function on* $[-1, 1]$. *Then, for n sufficiently large, there exists $P_n \in \Pi_n$ satisfying*

$$P_n \textit{ is comonotone with } f \textit{ on } [-1, 1]. \tag{8}$$

$$\|f - P_n\| \leqslant c_1\omega(f; n^{-1}), \tag{9}$$

where c_1 is a constant depending on properties of f.

In [4], we obtained certain estimates on the $D - 1$ kernel. These proofs may easily be modified to yield the following estimates for the general $D - j$ kernel, $V_{n,j}(t)$, defined by (3), and its derivative.

LEMMA 1. *Let $\epsilon > 0$ be given, and let $1 > \nu > 0$ be specified. Then there exist positive constants $d(j)$ and $d'(j)$, depending only on ν and j, such that, for all $n \geqslant 2(j+1)/\epsilon$,*

(i) *if $\nu \geqslant |x| \geqslant \epsilon$, then*

$$0 \leqslant V_{n,j}(x) \leqslant d(j)\, n^{-4j+1}\epsilon^{-4j}$$

and

(ii) *if $\nu \geqslant |x| \geqslant \epsilon$, then*

$$|V'_{n,j}(x)| \leqslant d'(j)\, n^{-4j+2}\epsilon^{-4j-2}.$$

Proof. (i) See [4].

(ii) We sketch the proof of (ii) for the $D - 1$ kernel. An easy modification yields the more general result:

Differentiating $V_{n,1}(x)$ by the quotient rule, and using the estimates on $\alpha_{1,n}$ given in [4], we get, letting $z(x) = (x^2 - \alpha_{1,n}^2)^2$, that

$$|(z(x))^{-1}| \leqslant 2\epsilon^{-4} \text{ for } n \geqslant 4/\epsilon,$$

and

$$|z'(x)/z(x)| \leqslant 3\epsilon^{-2} \text{ for } n \geqslant 4/\epsilon.$$

Now, using the estimates on $P_{2n}(x)$, $P'_{2n}(x)$, and $v_{n,1}$ given in [4] in the formula for $V'_{n,1}(x)$, we get (ii).

We will also need the following three lemmas:

LEMMA 2. *Let $j \in N$ be fixed. Let $V_{n,j}(t) = U_n(t)$ be the $D - j$ kernel defined by (3). Let $\mu \in [-\frac{1}{2}, \frac{1}{2}]$, and let $r_{n,j}(x - \mu) = r_n(x - \mu)$ denote the nth $D - j$ olynomial of the function $|x - \mu|$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$. Then*

$$r''_n(x-\mu) = 2U_n(x-\mu) - U_n(x-\tfrac{1}{2}) - U_n(x+\tfrac{1}{2}) + \tfrac{1}{2}U'_n(x+\tfrac{1}{2}) - \tfrac{1}{2}U'_n(x-\tfrac{1}{2}) + \mu(U'_n(x-\tfrac{1}{2}) + U'_n(x+\tfrac{1}{2})).$$

Proof. By definition we may write

$$r_n(x-\mu) = \int_{-1/2}^{1/2} U_n(t-x)|t-\mu|\, dt.$$

Keeping in mind that $U_n(t)$ is even, then making the substitution $u = x - t$ in the above integral and splitting up the new integral, we get

$$\begin{aligned} r_n(x-\mu) &= \int_{x-1/2}^{x+1/2} U_n(t) \, | x - \mu - t | \, dt \\ &= \int_{x-1/2}^{x-\mu} U_n(t)(x-\mu-t)\, dt - \int_{x-\mu}^{x+1/2} U_n(t)(x-\mu-t)\, dt. \end{aligned}$$

Differentiating this expression twice with respect to x yields our result.

We also get the following expression for the derivative of the nth $D - j$ polynomial of a function f:

LEMMA 3. *If $f \in C[-\frac{1}{2}, \frac{1}{2}]$ and $Q_{n,j}(f; x) = Q_n(x)$ denotes the nth $D - j$ polynomial of f given by* (4), *then*

$$\begin{aligned} Q_n'(x) &= -f(\tfrac{1}{2})\, U_n(x - \tfrac{1}{2}) + f(-\tfrac{1}{2})\, U_n(x + \tfrac{1}{2}) \\ &\quad + \int_{-1/2}^{1/2} U_n(t - x)\, df(t). \end{aligned} \tag{10}$$

Proof. As shown in [2] for the $D - 1$ polynomials of f, (10) follows by first integrating $Q_n(x)$ by parts, then differentiating with respect to x.

Note 3. Since we will be dealing primarily with the $D - 2$ kernel, we will adhere to the following notation: We denote the $D - 2$ kernel, $V_{n,2}(x)$, by $U_n(x)$, we let u_n be the normalizing constant $v_{n,2}$, and, if $\mu \in [-\frac{1}{2}, \frac{1}{2}]$, we denote by $r_n(x - \mu)$ the nth $D - 2$ polynomial of the function $| x - \mu |$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

It is well known that the convolution of a kernel with a continuous function f frequently approximates f "better" in places where f enjoys greater smoothness. We prove a spacial case of this general statement:

LEMMA 4. *Let f be a properly piecewise monotone function on $[-\frac{1}{2}, \frac{1}{2}]$, with local extrema at the points $\{\xi_i\}$ $(i = 1,..., p)$, where*

$$-\tfrac{3}{8} < \xi_1 < \xi_2 < \cdots < \xi_p < -\tfrac{1}{4},$$

and $p \geqslant 1$. Let f satisfy Definition 1 *with $\epsilon > 0$, and suppose, without loss of generality, that $\epsilon < d$, where d is taken with respect to the set $\{-\frac{3}{8}, -\frac{1}{4}, \xi_i\}$. Let M_i, m_i, and ϵ_i be defined as in Note* 2. *Let $Q_n(f; x)$ denote the $D - 2$ polynomial of f. Then, for each $i = 1,..., p$, there exists N_i such that if $x \in (\xi_i - \frac{1}{2}\epsilon_i, \xi_i + \frac{1}{2}\epsilon_i)$ and $n \geqslant N_i$,*

$$| f(x) - Q_n(f; x) | \leqslant d_2 M_i n^{-1}.$$

Proof. Let $a_i = \xi_i - \epsilon_i$, $b_i = \xi_i + \epsilon_i$, $i = 1,..., p$. Define the function $g(x)$ on $[-\frac{1}{2}, \frac{1}{2}]$ as follows:

$$\begin{aligned} g(x) &= f(a_i), & -\tfrac{1}{2} \leqslant x \leqslant a_i, \\ &= f(x), & a_i < x < b_i, \\ &= f(b_i), & b_i \leqslant x \leqslant \tfrac{1}{2}. \end{aligned}$$

Then $g(x) \in \mathrm{Lip}(1, 2M_i)$ by Note 2(i). Let $Q_n(g; x)$ denote the $D - 2$ polynomial of $g(x)$. Then if $x \in (\xi_i - \frac{1}{2}\epsilon_i, \xi_i + \frac{1}{2}\epsilon_i)$,

$$\begin{aligned} | f(x) - Q_n(f; x)| &= | g(x) - Q_n(f; x)| \\ &\leqslant | g(x) - Q_n(g; x)| + | Q_n(g; x) - Q_n(f; x)|, \end{aligned} \tag{11}$$

where, by our introductory remarks (see (5)),

$$\begin{aligned} | g(x) - Q_n(g; x)| &\leqslant \| g - Q_n(g)\|_{[-3/8, 1/4]} \\ &\leqslant 2c_1 M_i n^{-1}, \end{aligned} \tag{12}$$

and, by Lemma 1(i) (with $j = 2$),

$$\begin{aligned} &| Q_n(g; x) - Q_n(f; x)| \\ &\leqslant \left| \int_{-1/2}^{a_i} + \int_{b_i}^{1/2} [U_n(x - t)(g(t) - f(t))] \, dt \right| \\ &\leqslant d_1 \| f \| n^{-7}(\tfrac{1}{2}\epsilon_i)^{-8}, \end{aligned} \tag{13}$$

whenever $n \geqslant 12/\epsilon_i$.

For n so large that $n \geqslant 12/\epsilon_i$ and

$$d_1 \| f \| n^{-1}(\tfrac{1}{2}\epsilon_i)^{-1} \leqslant M_i \tag{14}$$

the result follows from (11)–(13).

3. Proof of the Main Result

Note. A piecewise linear function $L(x)$ defined on an interval $[a, b]$ is a (continuous) function for which there exist points $a = x_0 < x_1 < \cdots < x_k = b$, called the *nodes* of $L(x)$, such that $L(x)$ is linear on each of the intervals $[x_j, x_{j+1}]$, $j = 0, 1,..., k - 1$.

If $L(x)$ is a piecewise linear function with nodes at the points $x_0 < x_1 < \cdots < x_k$, we define S_j to be the slope of L on $[x_j, x_{j+1}]$, $j = 0, 1,..., k - 1$, and $M(L) = \max_j | S_j |$.

Proof of the theorem. Without loss of generality we will show that there exists a sequence of polynomials $\{P_n\}$ belonging to Π_{4pn} (where p is the number of local extrema) for n sufficiently large, and satisfying (8) and (9) for n sufficiently large. Again, without loss of generality, we will work on the interval $[-\frac{3}{8}, -\frac{1}{4}]$; unless otherwise indicated, all norms will be taken on this interval. On those occasions when it is necessary to choose a constant or an n which is sufficiently large, for the sake of simplicity, we will not always make the most economical choices.

Case 1. Suppose that $f(x) = L(x)$ is a piecewise linear function satisfying the hypotheses of the theorem. We note that because it is piecewise linear, $L(x)$ aytomatically satisfies a convexity condition for some $\epsilon > 0$. Let the nodes of $L(x)$ occur at the points

$$-\tfrac{3}{8} = x_1 < x_2 < \cdots < x_k = -\tfrac{1}{4},$$

and let

$$-\tfrac{3}{8} < \xi_1 < \xi_2 < \cdots < \xi_p < -\tfrac{1}{4}$$

be the local maxima and minima of L, where $p \geq 1$. Take d with respect to the set $\{-\frac{3}{8}, -\frac{1}{4}, \xi_i\}$ $(i = 1,..., p)$. Without loss of generality, we assume that $\epsilon < d$. We note that, since each $\xi_i = x_t$ for some $2 \leqslant t \leqslant k - 1$, then, using the notation of Note 2,

$$M_i = \max\{|S_{t-1}|, |S_t|\};\ m_i = \min\{|S_{t-1}|, |S_t|\}.$$

Let $0 < \epsilon_i < \epsilon$, $i = 1,..., p$, be such that (6) is satisfied for L. (Clearly, we may always take

$$\epsilon_i = \min\{(x_{t+1} - x_t), (x_t - x_{t-1})\}.)$$

We extend $L(x)$ comonotonely to $[-\frac{1}{2}, \frac{1}{2}]$ (and we denote the extension by $L(x)$ also), by defining

$$L(x) = L(-\tfrac{3}{8}) + S_1(x + \tfrac{3}{8}), \qquad -\tfrac{1}{2} \leqslant x \leqslant -\tfrac{3}{8},$$

and

$$L(x) = L(-\tfrac{1}{4}) + S_{k-1}(x + \tfrac{1}{4}), \qquad -\tfrac{1}{4} \leqslant x \leqslant \tfrac{1}{2}.$$

Defining $x_0 = -\frac{1}{2}$ and $x_{k+1} = \frac{1}{2}$, we may write

$$L(x) = A + \sum_{j=0}^{k} a_j\,|x - x_j|,$$

where A is a constant, and where

$$a_0 = (S_0 + S_k)/2,\ a_j = (S_j - S_{j-1})/2, \qquad j = 1,..., k.$$

Letting $Q_n(x)$ denote the $D - 2$ polynomial of the (extended) function $L(x)$, then, since

$$\| Q_{n,2}(A; x) - A \|_{[-3/8,-1/4]} = O(n^{-6}),$$

we may assume, without loss of generality, that

$$Q_n(x) = A + \sum_{j=0}^{k} a_j r_n(x - x_j). \tag{15}$$

We know that

$$\| L - Q_n \| \leqslant c_1 M(L)\, n^{-1}. \tag{16}$$

Define $\xi_0 = -\frac{7}{16}$ and $\xi_{p+1} = -\frac{3}{16}$. The remainder of the proof (for Case 1) will go as follows:

We will modify the sequence of polynomials $\{Q_n(x)\}$ in such a way as to obtain a sequence of polynomials $\{S_n(x)\}$ satisfying

$$S_n(x) \in \Pi_{4n-8} \text{ for each } n > p + 1. \tag{17}$$

For n sufficiently large,

$$\| L - S_n \| \leqslant d_3 M(L)\, n^{-1}$$

for some absolute constant d_3. (18)

There exists $d > \tau > 0$ such that for n sufficiently large, $S_n(x)$ is convex on the intervals $(\xi_i - \tau, \xi_i + \tau)$ for which ξ_i is a local minimum, concave on the intervals $(\xi_i - \tau, \xi_i + \tau)$ for which ξ_i is a local maximum, and comonotone on S_τ (see Note at beginning of Section 3). (19)

Finally, we will perturb the $S_n(x)$ to obtain the desired polynomials.

If we differentiate the expression (15) twice, and use Lemmas 1 ((i) and (ii)) and 2 (with $j = 2$), together with the fact that L satisfies a convexity condition for $\epsilon > 0$, to estimate separately each of the terms $a_j r_n''(x - x_j)$, we may obtain the following estimates for $Q_n''(x)$ in each of the intervals $(\xi_i - \frac{1}{2}\epsilon, \xi_i + \frac{1}{2}\epsilon)$, $i = 1,..., p$. We employ here the same technique that was used in [4] to estimate the second derivatives of the DeVore polynomials of a given piecewise linear function, and, omitting the details, we may assert the existence of a constant d_4 such that for all x belonging to one of these intervals for some $i = 1,..., p$, and for all $n \geqslant 12/\epsilon$,

$$Q_n''(x) \geqslant -(k + 1)\, d_4 M(L)\, n^{-6}\epsilon^{-8} \text{ for } \xi_i \text{ a local minimum,} \tag{20}$$

and

$$Q_n''(x) \leqslant (k + 1)\, d_4 M(L)\, n^{-6}\epsilon^{-8} \text{ for } \xi_i \text{ a local maximum.} \tag{21}$$

We now estimate $Q_n'(x)$ in the intervals $(\xi_i + h, \xi_{i+1} - h)$, $i = 0, 1,..., p$, where $0 < h < d$. Suppose that $x \in (\xi_i + h, \xi_{i+1} - h)$, where, without loss of generality, we assume that f is nondecreasing in (ξ_i, ξ_{i+1}). Using Lemma 3, with $f(x) \equiv L(x)$ and $j = 2$, $Q_n(x)$ is given by (10). Now

$$\begin{aligned}
&\int_{-1/2}^{1/2} U_n(t - x)\, dL(t) \\
&\quad = \left[\int_{-1/2}^{\xi_i} + \int_{\xi_{i+1}}^{1/2} U_n(t - x)\, dL(t)\right] + \int_{\xi_i}^{\xi_{i+1}} U_n(t - x)\, dL(t) \\
&\quad \geqslant - \left| \int_{-1/2}^{\xi_i} + \int_{\xi_{i+1}}^{1/2} U_n(t - x)\, dL(t) \right| \\
&\quad \geqslant -d_5 M(L)\, n^{-7} h^{-8}, \qquad \text{for all } n \geqslant 6/h, \qquad (22)
\end{aligned}$$

where, in the last inequality, we use Lemma 1(i) together with the fact that $\mathrm{Var}_{-1/2 \leqslant x \leqslant 1/2}(L) \leqslant M(L)$. We may assume, without loss of generality, that $L(-\frac{3}{8}) = 0$; hence, $\| L \|_{[-1/2, 1/2]} \leqslant M(L)$. Then, using Lemma 1(i), since $-\frac{7}{16} \leqslant x \leqslant -\frac{3}{16}$, it follows that

$$| -L(\tfrac{1}{2})\, U_n(x - \tfrac{1}{2}) + L(-\tfrac{1}{2})\, U_n(x + \tfrac{1}{2})| \leqslant d_6 M(L)\, n^{-7}, \qquad (23)$$

for each n.

Hence, from (10) and (22)–(23), we have that for $n \geqslant 6/h$ and $x \in (\xi_i + h, \xi_{i+1} - h)$, $i = 0, 1,..., p$,

$$Q_n'(x) \geqslant -d_7 M(L)\, n^{-7} h^{-8}, \text{ if } L \text{ is nondecreasing in } (\xi_i, \xi_{i+1}), \qquad (24)$$

and

$$Q_n'(x) \leqslant d_7 M(L)\, n^{-7} h^{-8}, \text{ if } L \text{ is nonincreasing in } (\xi_i, \xi_{i+1}), \qquad (25)$$

where we may take $d_7 = d_5 + d_6$.

We assume, without loss of generality, that p is odd and that $S_1 < 0$. We define

$$P(x) = \prod_{i=1}^{p} (x - \xi_i).$$

Then $\int_{-1/2}^{x} P(t)\, dt$ is comonotone with $L(x)$ on $[-\frac{1}{2}, \frac{1}{2}]$, and, for each $i = 1,..., p$,

$$P'(\xi_i) \geqslant d^{p-1} \text{ if } \xi_i \text{ is a local minimum } (i \text{ is odd}),$$

and

$$P'(\xi_i) \leqslant -d^{p-1} \text{ if } \xi_i \text{ is a local maximum } (i \text{ is even}).$$

By continuity, there exists $\tau' > 0$ (depending only on p and d) such that

$$P'(x) \geqslant \tfrac{1}{2}d^{p-1} \text{ in the intervals } (\xi_i - \tau', \xi_i + \tau'), \text{ if } i \text{ is odd, and}$$
$$P'(x) \leqslant -\tfrac{1}{2}d^{p-1} \text{ in the intervals } (\xi_i - \tau', \xi_i + \tau'), \text{ if } i \text{ is even.} \quad (26)$$

Let $\tau = \min_i\{\tau', \frac{1}{2}\epsilon_i\}$. Define

$$S_n(x) = Q_n(x) + \gamma_n \int_{-1/2}^{x} P(t)\, dt,$$

where

$$\gamma_n = d_8 M(L)\, n^{-5}\tau^{-8}d^{-p+1}, \quad (27)$$

(where we may take $d_8 > 2 \max\{d_4, d_7\}$). Then, clearly, $S_n(x)$ satisfies (17). It follows from (20)–(21), (24)–(25) (with $h = \tau$), and (26) that $S_n(x)$ satisfies (19) for all n such that

$$n \geqslant \max\{(k+1), 6/\tau\}. \quad (28)$$

Also, we note, using (27), that

$$\begin{aligned} | L(x) - S_n(x)| &\leqslant | L(x) - Q_n(x)| + \gamma_n \\ &\leqslant | L(x) - Q_n(x)| + d_8 M(L)\, n^{-5}\tau^{-8}d^{-p+1}, \end{aligned} \quad (29)$$

whenever $n > k$ and $x \in [-\frac{3}{8}, -\frac{1}{4}]$. By (19), we know that for n satisfying (28), $S_n(x)$ has exactly one peak in each interval $(\xi_i - \tau, \xi_i + \tau)$, $i = 1,..., p$. Let these peaks be $\{\xi_{i,n}^* = \xi_i^*\}$, $i = 1,..., p$, where

$$\xi_1^* < \xi_2^* < \cdots < \xi_p^* .$$

We make the following

Claim. For n sufficiently large,

$$| \xi_i - \xi_{i,n}^* | \leqslant d_{10} r n^{-1}, \qquad i = 1,..., p, \quad (30)$$

where $r = \max_i \{M_i/m_i\}$.

Proof of claim. Suppose that $x \in (\xi_i - \tau, \xi_i + \tau)$ and that n satisfies

$$n \geqslant N_i' \geqslant \max \left\{(k+1), 6/\tau, N_i, \left[\frac{M(L)\, \tau^{-8}d^{-p+1}}{M_i}\right]^{1/4}\right\}. \quad (31)$$

Then, using (29) together with Lemma 4, we have that

$$\begin{aligned} | L(x) - S_n(x)| &\leqslant d_9 M_i n^{-1} \\ &= e_{i,n} = e_i . \end{aligned} \quad (32)$$

We suppose, without loss of generality, that $\xi_i^* < \xi_i$ and that f is non-decreasing in (ξ_{i-1}, ξ_i). Then, using (7) (with $f = L$), together with (32), we have

$$\begin{aligned} S_n(\xi_i) + e_i \geqslant L(\xi_i) &\geqslant L(\xi_i^*) + m_i(\xi_i - \xi_i^*) \\ &\geqslant S_n(\xi_i^*) - e_i + m_i(\xi_i - \xi_i^*) \\ &\geqslant S_n(\xi_i) - e_i + m_i(\xi_i - \xi_i^*), \end{aligned}$$

from which (30) follows. Let $N' = \max_i \{N_i'\}$. We note that for $n \geqslant N'$, $S_n(x)$ satisfies (18).

We now perturb the $S_n(x)$ to obtain the desired polynomials. The technique is the same as the one used in [5, Theorem 1].

For $n \geqslant N'$, we let $w_n(x) = w(x)$ be the LaGrange Interpolating Polynomial of degree $p - 1$ such that

$$w(\xi_i) = \xi_i^*, \qquad i = 1,..., p.$$

Then we can write

$$w(x) = \sum_{i=1}^{p} \xi_i^* H_i(x),$$

where

$$H_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{p} \frac{x - \xi_j}{\xi_i - \xi_j}, \qquad i = 1,..., p.$$

Then

$$\begin{aligned} w(x) &= \sum_{i=1}^{p} (\xi_i + \eta_i) H_i(x) \\ &= x + \Sigma_i \eta_i H_i(x), \end{aligned}$$

where

$$| \eta_i | \leqslant h_n = d_{10} r n^{-1}. \tag{33}$$

Thus,

$$\| w(x) - x \|_{[-1/2, 1/2]} \leqslant a_p h_n ,$$

where

$$a_p = \max_{-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}} \sum_{i=1}^{p} | H_i(x) | .$$

Also,

$$w'(x) = 1 + \sum_{i=1}^{p} \eta_i H_i'(x),$$

so that

$$w'(x) \geqslant 1 - b_p h_n\,,$$

where

$$b_p = \max_{-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}} \sum_{i=1}^{p} |\, H_i'(x)|\,.$$

Hence, $w'(x) \geqslant 0$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$ if n satisfies

$$n \geqslant N'' \geqslant d_{10} r b_p\,. \tag{34}$$

Let $N = \max\{N', N''\}$. Define, for $n \geqslant N$,

$$P_n(x) = S_n(w(x)).$$

Then $P_n(x) \in \Pi_{4pn}$ with

$$P_n'(x) = S_n'(w(x))\, w'(x).$$

Hence,

$$\operatorname{sgn} P_n'(x) = \operatorname{sgn} S_n'(w(x))$$

for all $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $n \geqslant N$. Thus, $P_n(x)$ is comonotone with $L(x)$ on $[-\frac{3}{8}, -\frac{1}{4}]$ for all $n \geqslant N$.

Finally, taking norms on the interval $[-\frac{3}{8}, -\frac{1}{4}]$,

$$\| L - P_n \| \leqslant \| L - S_n \| + \| S_n(x) - S_n(w(x)) \|\,.$$

Now

$$\begin{aligned} | S_n(x) - S_n(w(x))| &\leqslant \omega(S_n\,;\, | w(x) - x |) \\ &\leqslant \omega(S_n\,;\, a_p h_n), \end{aligned}$$

where

$$\begin{aligned} \omega(S_n; h) &= \sup_{|x-y| \leqslant h} | S_n(x) - S_n(y)| \\ &\leqslant \sup\,[| S_n(x) - L(x)| + | L(x) - L(y)| + | L(y) - S_n(y)|] \\ &\leqslant 2 \| L - S_n \| + M(L)\, h. \end{aligned}$$

Thus, by (18) and (33), we have, for $n \geqslant N$,

$$\begin{aligned} \| L - P_n \| &\leqslant 3 d_3 M(L)\, n^{-1} + d_{10} a_p r M(L)\, n^{-1} \\ &= d_L M(L)\, n^{-1}, \end{aligned}$$

where the constant d_L depends on L.

Case 2. Suppose that f is an arbitrary function satisfying the hypotheses of the theorem. Let $f(x)$ have its peaks at the points

$$-\tfrac{3}{8} = \xi_0 < \xi_1 < \cdots < \xi_{p+1} = -\tfrac{1}{4},$$

where $p \geqslant 1$. We suppose, without loss of generality, that $f(-\tfrac{3}{8}) = 0$. We let M_i, m_i, and $0 < \epsilon_i < \epsilon$ have the same meanings they had in Note 2, where $f(x)$ satisfies Definition 1 with $0 < \epsilon < d$, where d is taken with respect to the set $\{\xi_i\}$ $(i = 0, 1,..., p + 1)$. Let $L_n(x)$ be the piecewise linear function whose nodes occur at all points

$$\xi_i + j/n \leqslant (\xi_i + \xi_{i+1})/2, j \in N^+, \qquad i = 0, 1,..., p,$$

and at all points

$$\xi_i - j/n \geqslant (\xi_{i-1} + \xi_i)/2, j \in N^+, \qquad i = 1,..., p + 1,$$

with the exception of those nonpeaks which have distance less than $1/n$ from each other. Let the nodes of $L_n(x)$ be the points $\{x_j\}$ $(j = 0, 1,..., s - 1)$, where

$$-\tfrac{3}{8} = x_0 < x_1 < \cdots < x_{1-1} = -\tfrac{1}{4}.$$

We define $L_n(x_j) = f(x_j)$, $j = 0, 1,..., s - 1$. Then $L_n(x)$ has the following properties:

(i) $1/n \leqslant x_{j+1} - x_j \leqslant 4/n, j = 0, 1,..., s - 2.$

(ii) $\| f - L_n \| \leqslant \omega(f; 4/n) \leqslant 4\omega(f; n^{-1}).$ (35)

(iii) L_n is comonotone with f on $[-\tfrac{3}{8}, -\tfrac{1}{4}]$.

(iv) $M(L_n) \leqslant 4n\omega(f; n^{-1}).$ (36)

(v) For $n \geqslant 6/\epsilon$, the slopes of $L_n(x)$ are increasing in the intervals $(\xi_i - \tfrac{1}{2}\epsilon, \xi_i + \tfrac{1}{2}\epsilon)$ for which ξ_i is a local minimum, and decreasing in the intervals $(\xi_i - \tfrac{1}{2}\epsilon, \xi_i + \tfrac{1}{2}\epsilon)$ for which ξ_i is a local maximum.

(vi) For $n \geqslant 6/\epsilon_i$, $| S_j | \leqslant 2M_i$ for all $x_j \in (\xi_i - \tfrac{1}{2}\epsilon_i, \xi_i + \tfrac{1}{2}\epsilon_i)$.

(vii) For $n \geqslant 6/\epsilon$, $| S_j | \geqslant m_i$ for all $x_j \in (\xi_i - \tfrac{1}{2}\epsilon, \xi_i + \tfrac{1}{2}\epsilon)$.

(viii) $\| f \| = \| L_n \|$ for each n.

(ix) $s + 1 \leqslant n$ if $n \leqslant 2p + 4$.

Now, by Case 1, for all n satisfying (34) and satisfying (14) and (31), where we let $\tfrac{1}{2}\epsilon_i$, s, $M(L_n)$, and $\| L_n \|$ play the roles of ϵ_i, k, $M(L)$, and $\| f \|$, respectively, $i = 1,..., p$, and where we take $\tau = \min_i \{\tau', \tfrac{1}{4}\epsilon_i\}$, we can find

$P_n(x) \in \Pi_{4pn}$ such that $P_n(x)$ is comonotone with L_{npn} such that $P_n(x)$ is comonotone with $L_n(x)$, hence with $f(x)$, on $[-\frac{3}{8}, -\frac{1}{4}]$, and such that

$$\| L_n - P_n \| \leqslant d_L M(L_n)\, n^{-1}, \tag{37}$$

where d_L depends on $r = \max_i \{M_i/m_i\}$ and on the peaks of f. Then, using (35)–(37) together with the fact that

$$\| f - P_n \| \leqslant \| f - L_n \| + \| L_n - P_n \|,$$

we have that

$$\| f - P_n \| \leqslant d_f \omega(f; n^{-1}),$$

where d_f depends on f. Q.E.D.

Remark. We note that we can derive the estimate

$$E_n^*(f) = o(\omega(f; n^{-1+\epsilon})), \qquad \epsilon > 0$$

(see (2)), for an arbitrary piecewise monotone function $f \in C[-1, 1]$ by modifying the proof of the theorem slightly. We will briefly sketch the means by which this can be done:

Step 1. Suppose that $\epsilon > 0$ is given, and we wish to approximate $f \in C[-\frac{3}{8}, -\frac{1}{4}]$ comonotonely by elements of Π_n with error of smaller order of magnitude than $O(\omega(f; n^{-1+\epsilon}))$. Choose $j \in N$ so that $j^{-1} < \epsilon$ and let $t = 1 - j^{-1}$. Approximate f by piecewise linear functions whose nodes are spaced at least n^{-t} apart and at most $4n^{-t}$ apart and which include the peaks of f. Let

$$-\tfrac{3}{8} = x_0 < x_1 < \cdots < x_k = -\tfrac{1}{4}$$

be the nodes of L_n and define $L_n(x_j) = f(x_j)$, $j = 0, 1, \ldots, k$. Then

(i) L_n is comonotone with f on $[-\frac{3}{8}, -\frac{1}{4}]$,

(ii) $\| f - L_n \| \leqslant 4\omega(f; n^{-t})$,

(iii) $M(L_n) \leqslant 4n^t \omega(f; n^{-t})$, and

(iv) L_n is convex [concave] in an interval of radius n^{-t} about each local minimum [maximum] of f.

Step 2. Extend $L_n(x)$ to $[-\frac{1}{2}, \frac{1}{2}]$ in the same way we extended $L(x)$ above. Let $Q_{n,j}(x)$ be the nth $D - j$ polynomial of $L_n(x)$ for each $n \geqslant j$. Using Lemmas 1–3 (with the j we selected in Step 1), we can construct polynomials $S_{n,j}(x) \in \Pi_{4(n-j)}$ in a manner similar to that in which the $S_n(x)$ were constructured above. The $S_{n,j}(x)$ can be shown to satisfy, for n sufficiently large, the estimate

$$\| L - S_{n,j} \|_{[-3/8,-1/4]} \leqslant d_{11} M(L_n)\, n^{-1}$$

for some constant d_{11} and the condition (19) where we may take $\tau = n^{-t}$.

Step 3. Noting that the $S_{n,j}(x)$, for n sufficiently large, have exactly one peak in neighborhoods of radius n^{-t} about each peak of f, we perturb the $S_{n,j}(x)$ to obtain polynomials $P_n(x)$, comonotone with f on $[-\frac{3}{8}, -\frac{1}{4}]$ and satisfying

$$\begin{aligned} \| f - P_n \| &= O(\omega(f; n^{-t})) \\ &= o(\omega(f; n^{-1+\epsilon})). \end{aligned}$$

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